SPOLM 2008

ISSN 2175-6295 Rio de Janeiro- Brasil, 05 e 06 de agosto de 2008.

A MODEL OF STOCK MANAGEMENT AND A MODEL OF A NON RENEWABLE NATURAL RESOURCES MANAGEMENT SOLVED BY DYNAMIC OPTIMISATION

Gerardo Oscar Bertoluzzo
Universidad Tecnológica Nacional
Facultad Regional Venado Tuerto
Argentina
gbertol@cablenet.com.ar

Jacinto Diab
Universidad Tecnológica Nacional
Facultad Regional Venado Tuerto
Argentina
jacintodiab@hotmail.com

Pablo Euclídes Baigorria
Universidad Tecnológica Nacional
Facultad Regional Venado Tuerto
Argentina
p_euclidesbaigorria@hotmail.com

Luis Víctor Barello
Universidad Tecnológica Nacional
Facultad Regional Venado Tuerto
Argentina
barello@coopsancti.com.ar

KEYWORDS

DINAMIC OPTIMISATION-STOCK MODEL-NON RENEWABLE RESOURCES

ABSTRACT

The Dynamic Optimisation studies the attainment of the optimum solutions of systems which valuate in time,susceptible of influence by means of external decisions.The best decision to take,depends on the temporal horizon from where we can see the problem to be studied.In general the optimum decision in a dynamical context can not be obtained by a sequence of static and optimum decisions for each one of the instants or periods that constitute the mentioned dynamic context.So that,the optimum decisions in short time,generally do not coincide with the optimum decisions in long term.The employment of Dynamic Optimisation Technics allows to get to the optimum solution in each case.In this paper through two concrete problems,the basic problem of variational calculus,has been formulated and two interpretations are gived.

A MODEL OF STOCK MANAGEMENT

In the t_0 instant a company has a $x(t_0) = x_0$ stock of a certain product.The support of each unit of such stock implies an instantaneus cost of c monetary units.The company can use the units it keeps in stock to sell them at certain market.The sale of q units in this market provides a benefit,not taking into account the cost of keeping the stock,which is given by function $\Pi(q)$,which is supposed to be concave.

The company wants to distribute its stock in an optimum way along a planified horizon $[t_0, t_1]$,so that its its total profis may be maximized along the period of time.It is assumed that all along that time it is not probable that he company may acquiere other goods units and that the units no sold in the given temporal horizon have no value after t_1 instant.

The problem to be faced is:

$$\max_{q,x} \int_{t_0}^{t_1} (\Pi(q) - cx)dt$$

with:

$$\dot{x} = -q; x(t_0) = x_0; x(t_1) \geq 0$$

Using the restriction to eliminate variable q,the previus problem can be expressed:

$$\max_{x} \int_{t_0}^{t_1} (\Pi(-\dot{x}) - cx)dt$$

with:

$$x(t_0) = x_0; x(t_1) \geq 0$$

For this problem:

$$F(x, \dot{x}, t) = \Pi(-\dot{x}) - cx$$

so:

$$F_x = -c; F_{\dot{x}} = -\Pi'$$

and Euler's equation is:

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0$$

that is to say:

$$c = \frac{d}{dt}\Pi' \qquad (1)$$

Equation (1) has the following interpretation:when deciding the amount of products taken from the stock,the company compares the profit and loss derived from the option of keeping the stock or selling it.The left side of (1) represents the marginal cost for keeping a product in stock,wich is the storage cost.The right size represents the profit derived from keeping the stock unit ,that is to say,how marginal benefits that could be obtained from that unit grow in time.

Legendre's condition for maximum, $\frac{\partial^2 F}{\partial \dot{x}^2} = \Pi'' \le o$;is fulfilled through hypothesis Π is a concave function(that fulfills $\Pi'' \le 0)$.

As the final value of the stock is restricted to be non negative $(x(t_1) \ge 0)$,the transversality condition is:

$$[F_{\dot{x}}]_{t=t_1} \le 0 \text{ ;(=0 if } x(t_1) > 0)$$

that is to say:

$$[\ \Pi']_{t=t_1} \ge 0; (= 0 \text{ if } x(t_1) > 0\ )$$

The interpretation of this condition is the following: in the last instant,there is no reason for keeping stock for futures sales,so,stock should be sold out ($x(t_1) = 0$),or up to a zero marginal benefit in no case should any product be taken out from stock causing a low rage in profits,that is to say: $\Pi' < 0$.

The sufficient condition is achieved if the function $F(x, \dot{x}, t)$ is concave in $(x, \dot{x})$ for each $t \in [t_0, t_1]$.

Is obtained:

$$\nabla_{x\dot{x}} F = (F_x, F_{\dot{x}})$$

and

$$H_{x,\dot{x}} F = \begin{pmatrix} F_{xx} & F_{x\dot{x}} \\ F_{\dot{x}} & F_{\dot{x}\dot{x}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Pi'' \end{pmatrix}$$

and this matrix is negative semidefined,since $\Pi'' \le 0$,because Π is a concave function.

Let us suppose that the company is allowed to sell in a short term,that is to say,get to sell more products than those stored.In such case, $(x(t_1) \ge 0\ )$ should no longer be demanded.So,the transversality condition is:

$$[F_{\dot{x}}]_{t=t_1} = 0$$

that is to say:

$$[\Pi']_{t=t_1} = 0$$

meaning that the right amount will be extracted,what makes its marginal profit zero.

If Π'>0,it is optimum to increase the sale of product going into,if necessary,an overdraft.On the contrary,if $\Pi' < 0$,sales should be reduced up to the point of $\Pi' = 0$.

A MODEL OF A NON RENEWABLE RESOURCE MANAGEMENT

A company has the exploitation rights of a non-renewable natural resource.Its benefit function is $\Pi(x, q, t)$ where x represents the existing resource stock in the t instant,and q represents the instantaneus extraction of the above mentioned resource.

The fact that the extraction rate appears in the profit function is easily understod.The presence of stock in such function can be justified by means of its influence on extraction cost:at a greater available stock,there is a lower extraction cost,and,in consequence,greater profits.

Another interpretation for the influence of x variable consist in adscribing two alternative uses to the resource:one of them as a productive resource by means of its extraction,and the other as an entertainment resource,which makes it valuable in its natural state.

The resource stock develops in time according to this differential equation:

$$\dot{x} = -q$$

so that the benefit function can be expressed as:

$$F(x, \dot{x}, t) = \Pi(x, -\dot{x}, t)$$

and

$$F_x = \Pi_x ; F_{\dot{x}} = -\Pi_q$$

is fulfilled.

The company must decide on a policy of extraction/conservation resources with the aim of maximizing its flow of benefits in a certain temporal horizon between t_0 and t_1.For that reason,it faces the following problem as regards variational calculus:

$$\max \int_{t_0}^{t_1} F(x, \dot{x}, t)dt$$

with:

$$x(t_0) = x_o ; x(t_1) \geq 0$$

Euler's equation is:

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0$$

that,in this case can be expressed as:

$$\Pi_x + \frac{d}{dt} \Pi_q = 0$$

This equation determines the stock that is worth while keeping at every moment.In fact,keeping the resource provides two types of profit;in the first place,an additional unit of x allows to obtain the additional inmediate marginal benefit of Π_x.On the other hand,the available stock can be extracted in the future,obtaining a Π_q marginal benefit.

The term: $\frac{d}{dt}\Pi_q$,measures the temporal change in such marginal benefits.When equation:

$$\Pi_x + \frac{d}{dt} \Pi_q = 0$$

the resource more valuable in its real state(that is to say,the soil deposit) so that it is profitable making use of the real stock and wait to exploit it in future time.When:

$$\Pi_x + \frac{d}{dt} \Pi_q < 0$$

waiting time produces the opposite effect,which justifies the total use of the stock as soon as possible.All along the optimum policy,at every moment,the right amount of resource is extracted in order to produce equal conditions,so that it is not possible to obtain profits extracting larger or smaller quantities.

Legendre's condition is: $F_{\dot{x}\dot{x}} \leq 0$ that equals : $\Pi_{qq} \leq 0$;that is to say,the benefit is a concave function in the resource extraction.

The transversality condition is:

$$[F_{\dot{x}}]_{t=t_1} \leq 0, (= 0 \text{ if } x(t_1) > 0)$$

that is to say:

$$[\Pi_q]_{t=t_1} \geq 0, (=0 \text{ if } x(t_1) > 0)$$

The interpretation is the following:in the last moment of the planning period,there is no reason to keep the resource,so the marginal benefit should be attended which provides its extraction.In fact,if the marginal benefit is positive ($\Pi_q > 0$) the resource should be exploited up to its exhaustion ($x(t_1) = 0$).The only reason for keeping a stock up to the end of the period without exploiting is that the marginal benefit may be just zero,so that a small increase in the extraction rate would not generate any increase in benefits.

At last,there will never be such a high extraction that would make benefits descended,that is to say $\Pi_q < 0$.If this situation happened,a lower extraction rate would be the right procedure,so that $\Pi_q = 0$.

A SPECIAL CASE

Let us suppose that the benefit function has this shape:

$$\Pi(x,q,t) = e^{-\delta t}(pq - C(x))$$

where δ is the intertemporal discount rate, p is the market price for the extracted resource and $C(x)$ represents the extraction cost.

Let us suppose that $C' < 0$,that is to say,the larger the stock of available resource,the less expensive is the extraction.The problem remains as:

$$\max \int_{t_0}^{t_1} e^{-\delta t}(pq - C(x))dt = \min \int_{t_0}^{t_1} e^{-\delta t}(p\dot{x} + C(x))dt$$

Now Euler's equation is:

$$e^{-\delta t}C'(x) = \frac{d}{dt}(e^{-\delta t}p) = e^{-\delta t}(-\delta p + \dot{p}) \Leftrightarrow C'(x) = -\delta p + \dot{p}$$

which can also be expressed as:

$$\delta p = \dot{p} - C'(x)$$

which is Hotelling's rule that expresses the balance between both alternatives:to extract or to keep the resource.The left side represents the marginal profit for extracting inmediately a resource unit:the price obtained for it multiplied by the intertemporal profit.The right side represents the marginal profit for waiting,increase in price(remember it is negative) plus the savings in the future costs in extraction owing to a greater stock.

Taking into account:

$$\frac{\partial^2 F}{\partial \dot{x}^2} = \frac{\partial^2 \Pi}{\partial q^2} = 0$$

the Legendre's condition is fullfilled,and now the transversality condition is:

$$p(t_1) \geq 0; (=0 \text{ if } x(t_1) > 0)$$

but if at the last moment the resource is not exhausted,the reason is that the price at that moment,is zero.

BIBLIOGRAPHY

1)"Ecuaciones Diferenciales y Cálculo Variacional";Lev Elsgoltz,Editorial Mir

2)"Cálculo Variacional",Krasnov,Kiseliov,Makarenko,Editorial Mir

3)"Técnicas Matemáticas de Investigación Operacional",L.S.Goddard,Editorial Exedra

4)"Optimización Dinámica"E.Cerdá,Editorial Prentice Hall

5)"Dynamic Optimization in Discrete Time"H.D.Stark,Illinois,University Press